Lecture 14

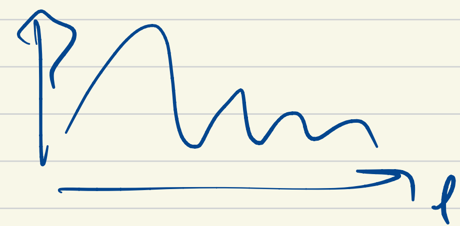
- Review
- Inflation in homogeneous limit (cont.)
- Free scalar field in de Sitter
- Inflationary perturbations   GR 13.1

advanced: 1609.00716

Review:

- Finished CMB perturbations

$\Phi \sim \Phi_i \cdot \cos c_s k \eta$



↳ initial conditions (primordial)

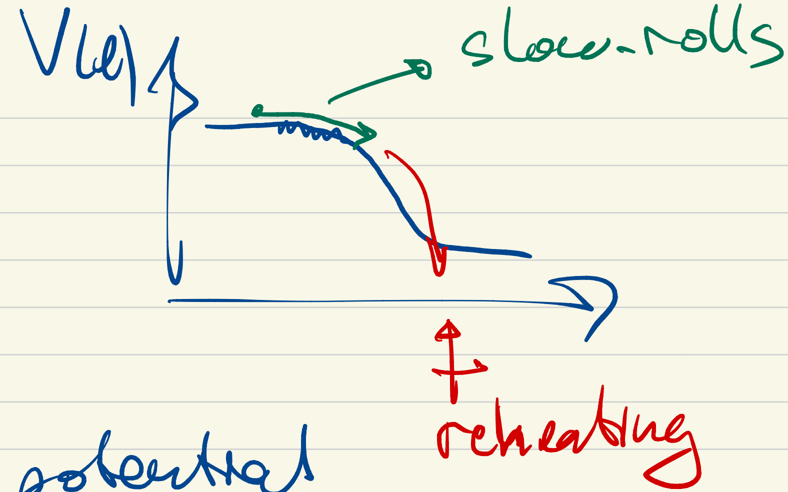
$\Phi_i(\vec{k})$ — random gaussian variable

$$\langle \Phi^i_{-\vec{k}_1} \Phi^i_{\vec{k}_2} \rangle = \delta(\vec{k}_1 - \vec{k}_2) P(|k|)$$

$$P(|k|) \sim 10^{-10} \left(\frac{k_*}{k}\right)^{3+(n_s-1)}$$

↳ where is it coming from?

Theory of inflation



- de Sitter space
- scalar field with a potential

$\varphi \to \varphi(t) + \varphi(x,t)$

$$S = \int d^4x \sqrt{g}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu \varphi \partial_\nu \varphi - V(\varphi)\right]$$

$\varphi = \varphi(t)$ $\qquad \ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0$

slow-roll approx.

$$\left|\frac{\ddot{\varphi}}{3H\dot{\varphi}}\right| << 1 \quad (**) \qquad H = \text{const} \qquad \frac{\dot{\varphi}^2}{2V(\varphi)} << 1 \quad (*)$$

$$\dot{\varphi} = -\frac{1}{3H} V' \qquad H = \frac{1}{M_{pl}} \left(\frac{8\pi V}{3} \right)^{1/2}$$

$$(*) \rightarrow \underbrace{\frac{M_{pl}^2}{16\pi} \left(\frac{V'}{V} \right)^2}_{\varepsilon} \ll 1$$

ε (slow-roll parameter)

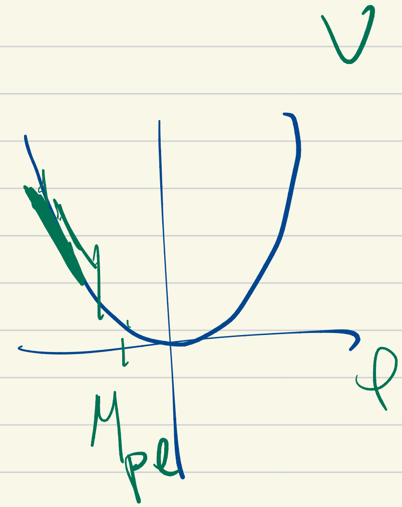
$(**)$ in terms of potentials

$$\ddot{\varphi} = M_{pl} \left(\frac{V''}{V^{1/2}} - \frac{1}{2} \frac{V'^2}{V^{3/2}} \right) \cdot \dot{\varphi} =$$

$$= M_{pl}^2 \left(\frac{V''}{V} - \cancel{\frac{1}{2} \left(\frac{V'}{V} \right)^2} \right) H \dot{\varphi}$$

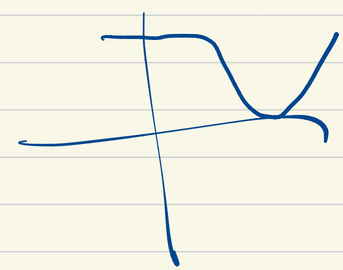
$$\frac{V''}{V} \ll \frac{24\pi}{M_{pl}^2}$$

$$\eta = \frac{M_{pl}^2}{8\pi}\frac{V''}{V} \ll 1$$



$\varepsilon, \eta \rightarrow$ slow roll parame

$$V = \frac{m^2}{2}\varphi^2 \quad \text{for slow-roll?}$$



$$\frac{M_{pl}^2}{\varphi^2} \ll 1 \qquad \frac{M_{pl}^2}{\varphi^2} \ll 1 \qquad m^2\varphi^2 \ll M_{pl}^4$$

• Perturbations

→ Free massless scalar field in Minkowski

$$S_\varphi = -\frac{1}{2}\int d^4x\, \eta^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi \qquad \pi = \dot{\varphi}$$

$$E = H = \frac{1}{2}\int d^3x\left[\dot{\varphi}^2 + (\partial_i\varphi)^2\right]$$

in momentum space decoupled harmonic oscillators

$$\varphi(x_i, t) \xrightarrow[\substack{\text{Fourier} \\ e^{i\vec{q}\cdot\vec{x}}}]{} \varphi(\vec{q}, t) \longrightarrow \hat{\varphi}(\vec{q}, t)$$

$$\varphi(\vec{x},t) \to \hat{\varphi}(\vec{x},t) = \int \frac{d^3 q}{(2\pi)^{3/2}\sqrt{2\omega_q}} \cdot$$

$$\cdot \left(e^{i\omega_q t - i\vec{q}\cdot\vec{x}} A_q^+ + e^{-i\omega_q t + i\vec{q}\cdot\vec{x}} A_q \right)$$

$$\omega_q = |\vec{q}| \qquad [A_q, A_{q'}^+] = \delta^3(\vec{q}-\vec{q}\,')$$

$$H = \int d^3 q\, \omega_q A_q^+ A_q$$

e.g. harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \qquad x = \frac{a + a^\dagger}{\sqrt{\omega}}$$

$$a = \sqrt{\frac{m\omega}{2}} \left(x + \frac{ip}{m\omega} \right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2}} \left(x - \frac{ip}{m\omega} \right)$$

$$H = \omega a a^\dagger \qquad [a, a^\dagger] = 1$$

In the ground state $\langle x \rangle = 0$

$$\langle x^2 \rangle = \frac{1}{m\omega} \neq 0$$

$$x \rightsquigarrow \varphi(\vec{q}) \sim \varphi_i$$

$$\langle \varphi_q \varphi_{q'} \rangle = \delta^3(q+q') \frac{1}{\omega_q} = \delta^3(q+q') \frac{1}{|q|}$$

(stays small ~ quantum)

Inflationary perturbations

- Full procedure: GR 13.1

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→ repeat steps of classical theory of perturbations for inflationary theory (fix the gauge, S–V–T decomposition)

→ quantize corresponding fields

- Simplified treatment:

  Just quantize scalar field on fixed grid → → de Sitter

  ~ turns out in leading order in ε, η it is correct

  ~ (nothing conceptually complicated to do full treatment)

- Inflaton field perturbations

$$\varphi(x,t) \rightarrow \text{Inflaton (scalar field)}$$

$$\varphi(x,t) = \varphi_{cl}(t) + \varphi(x,t)$$

$$S_\varphi = \frac{1}{2}\int d^4x \sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - \cancel{V''(\varphi_0)}\cdot\varphi^2\right)$$

$g^{00} \sim \varphi$ small $\sim \eta$

quadratic action $\Longleftrightarrow$ linear equations

EOM:

$$\ddot\varphi + 3H\dot\varphi - \frac{1}{a^2}\partial_i\partial_i\varphi$$

(in FRW time)

$-dt^2 + e^{2Ht}d\vec{x}^2$

$$\rightarrow \varphi'' + 2\frac{a'}{a}\varphi' - \Delta\varphi = 0$$

(in conf. time)

$\sim \frac{1}{|\eta|}k$ $\quad\sim k^2$

$a = \frac{1}{-\eta}$

$\frac{1}{\eta} k \gg k^2$ $|\eta k| \ll 1$ → outside horizon
(horizon scale)

$\frac{1}{\eta} k \ll k^2$ $|\eta k| \gg 1$ → inside horizon

$t \nearrow$ $|\eta| \searrow$ modes exit horizon

$t \to \infty$ (fixed k) $|\eta| \to \infty$
$(\eta \to -\infty)$

$\varphi'' - \Delta \varphi = 0$ ⇒ same as
in flat space
~ expect ground state osc.

$$\chi = a(\eta) \cdot \varphi$$

$$S_\chi = \frac{1}{2} \int d^3x \, d\eta \left(\chi'^2 - (\partial_i \chi)^2 + \frac{a''}{a} \chi^2 = \right.$$

go to k space

$$= \int \frac{1}{2} \frac{d^3k}{(2\pi)^3} d\eta \left[\chi'(\vec{k}) \chi'(-\vec{k}) - \left(k^2 + \frac{a''(\eta)}{a(\eta)} \right) \cdot \chi(\vec{k}) \chi(-\vec{k}) \right)$$

frequency depends on time

$$\chi(x,y) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k}} \left(e^{-ikx} \chi_k^{+}(y) A_k^{+} + e^{ikx} \chi_k^{(-)}(y) A_k \right)$$

$$[A_k, A_{k'}^{+}] = \delta(\vec{k}-\vec{k}')$$

what are χ^+ and χ^-

in flat space we had $\underbrace{e^{\pm i\omega_k y}}$

solutions of EOM $(\ddot{\varphi}_k = \omega^2 \varphi_k)$ $\omega_k = |\vec{k}|$

Quantum operators satisfy same EOM as classical fields

$$\left[\ \dot{p} = [H, x] = \omega^2 x \qquad \dot{x} = [H, p] = p\ \right]$$

$$\chi_k''^{\,cl} - \frac{2}{\eta^2}\chi_k^{cl} + k^2 \chi_k^{cl} = 0$$

$$\chi^{\pm} = e^{\pm i k \eta}\left(1 \pm \frac{i}{k\eta}\right)$$

here initial conditions are fixed by equating to flat space vacuum fluctuations at $|k\eta| \gg 1$

- What happens for $|k\eta| \ll 1$ (late times)?

$$\langle \varphi(k)\, \varphi(k') \rangle \sim \delta^3(\vec{k}+\vec{k}') \cdot$$

$$\cdot\ a^{-2}(\eta) \cdot \underbrace{\frac{1}{k}}_{\text{Mink.}} \frac{1}{|k\eta|^2} = \delta^3(k+k') \frac{H^2}{k^3}$$

Scale invariant power spectrum

vs $\frac{1}{k}$ in flat space

$$P \sim \left(\frac{H^*}{k}\right)^3$$

$$d^3k \, \frac{1}{k^3} \simeq \text{const} \longleftrightarrow \langle \varphi(x)\, \varphi(y) \rangle \sim$$

$$\langle x^2 \rangle$$

$$\sim \log(x-y)$$

- This is just fluctuations of inflaton field how do we go to grav. potential?

- First, note that state of superhorizon modes are (semi-)classical

$$\frac{1}{|k\eta|^2} \frac{1}{k} \gg \frac{1}{k}$$

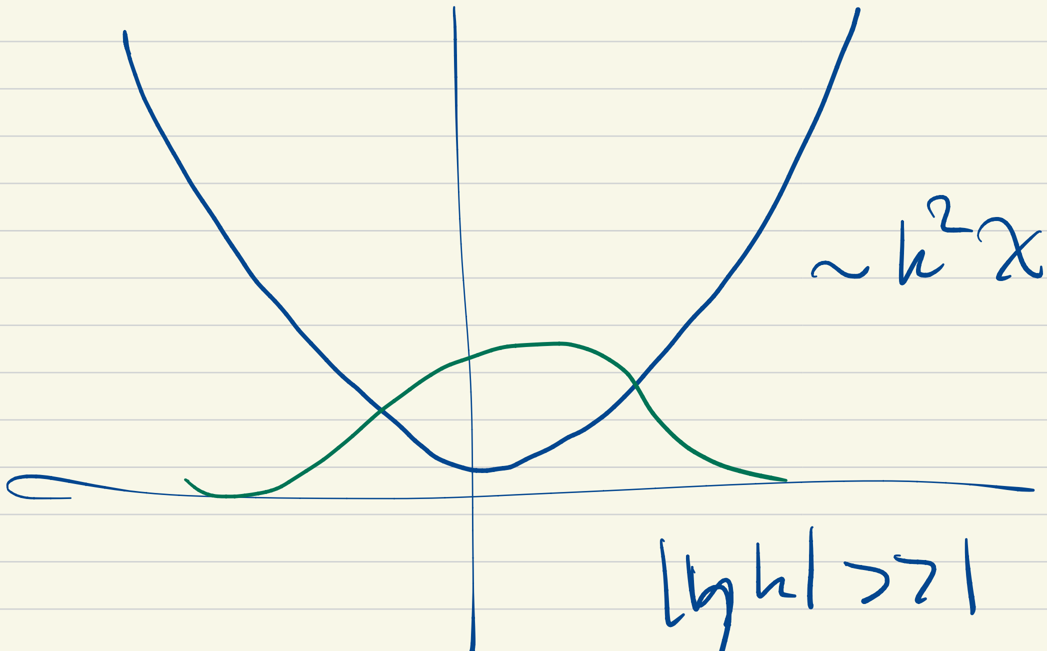
$$\langle \chi_k^2 \rangle \quad \text{c.f.} \quad \langle x^2 \rangle$$

$$\langle n|x^2|n\rangle \gg \langle 0|x^2|0\rangle = \frac{1}{\omega} = \frac{1}{k}$$

$$\Downarrow$$

$|n\rangle$ is semiclassical

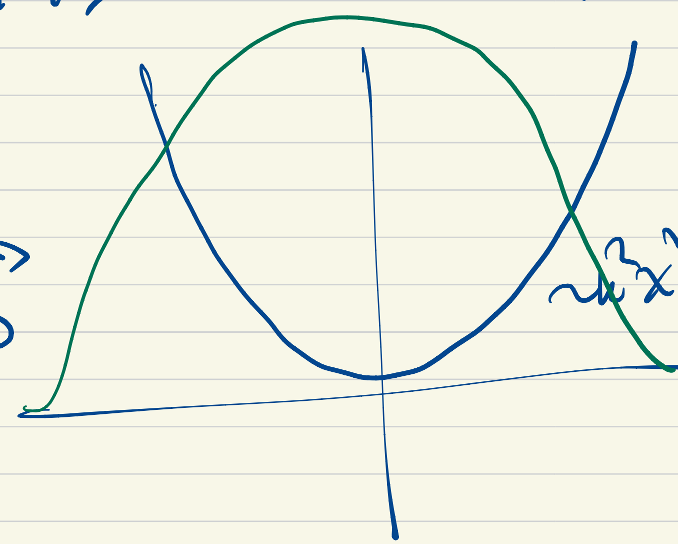
$|0\rangle$



$\sim k^2 x^2$

$|\log k| \gg 1$

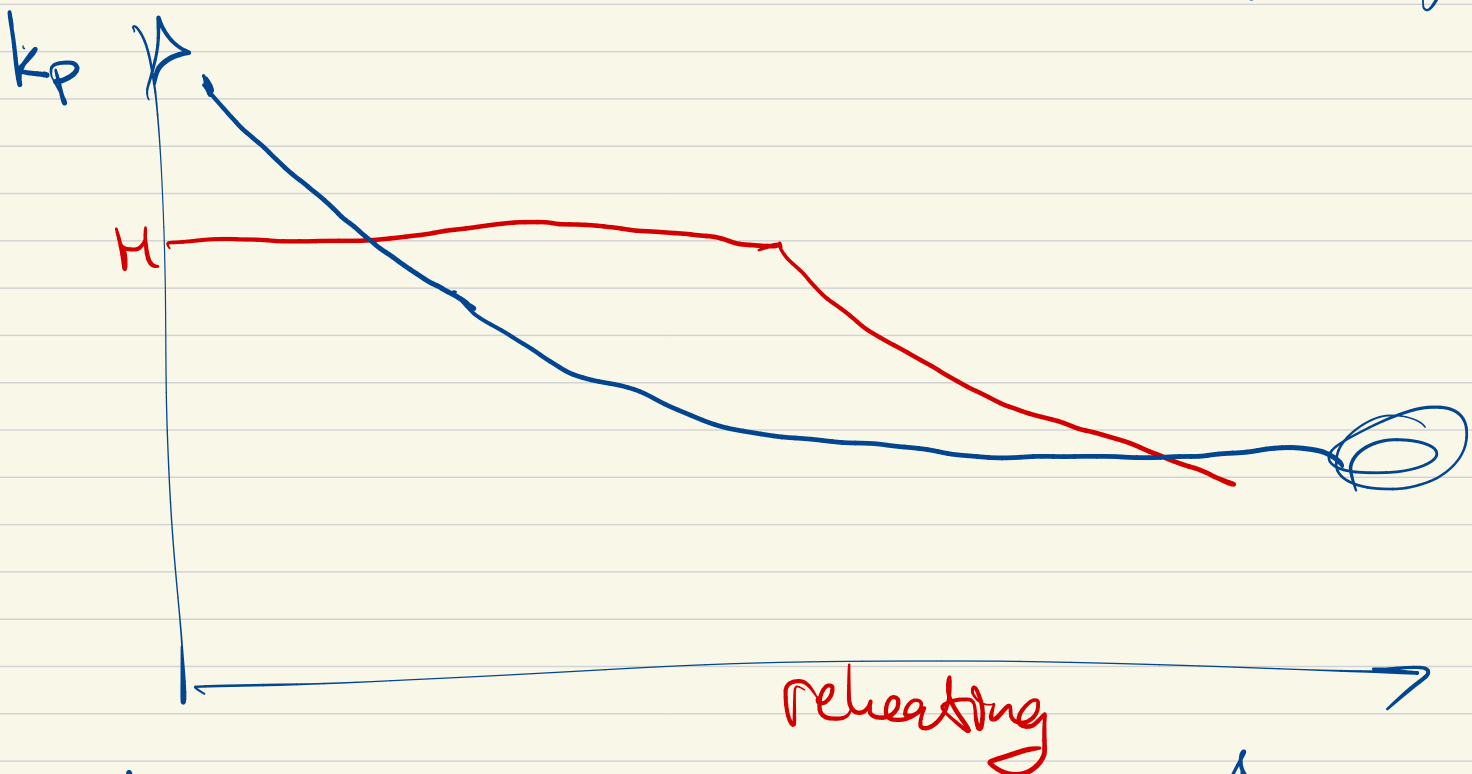
$\xrightarrow{\qquad}$ $|\log| \to 0$

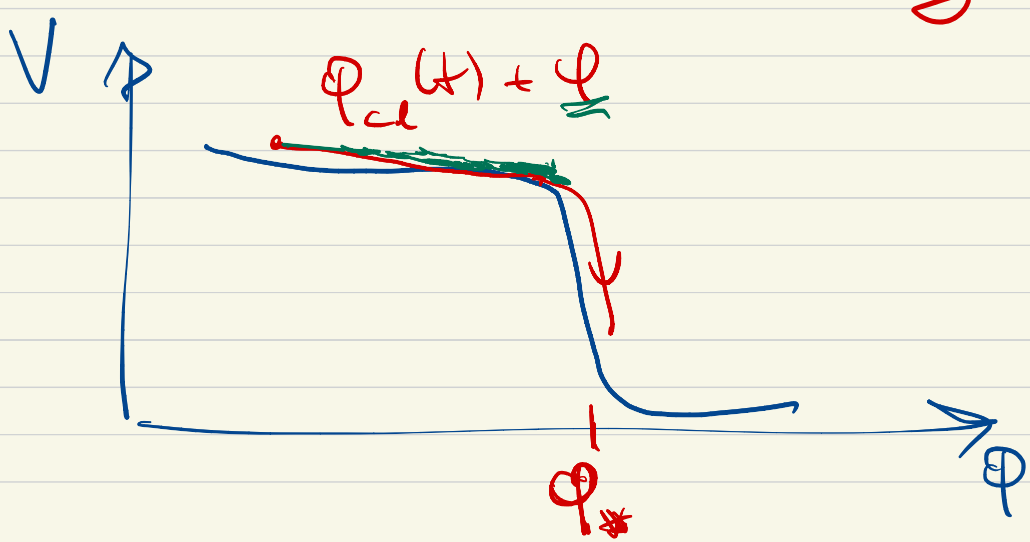


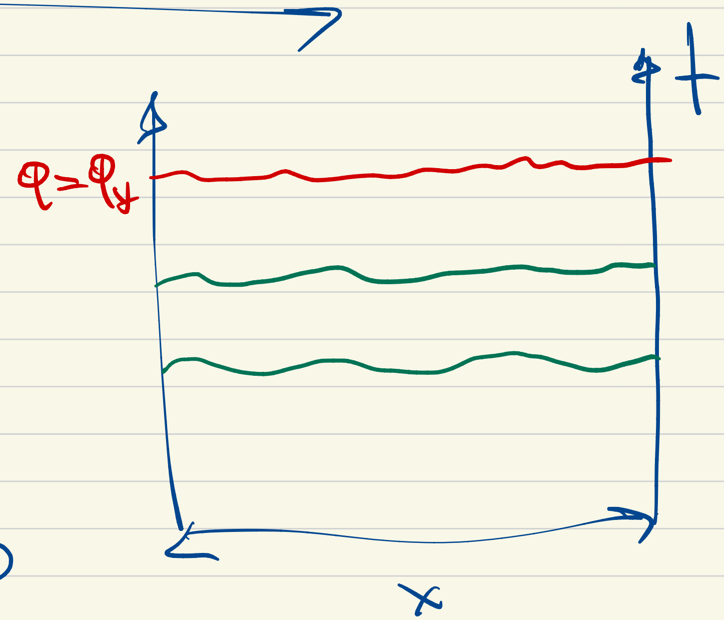
$\sim k^3 x^2$

(high occupation numbers)

it allows us to transition to classical description towards the end of inflation







$$\delta t_r(\vec{x}) = \frac{\varphi(\vec{x})}{\dot{\varphi}_{cl}} \qquad \left.\frac{\delta a}{a}\right|_{t_r} = \text{grav. pot. } (\Phi_i)$$

$$\Phi_i = \left.\frac{\delta a}{a}\right|_{t_r} = \frac{\dot{a}\cdot\frac{\varphi}{\dot{\varphi}_{cl}}}{a} = \frac{H\varphi}{\dot{\varphi}_{cl}}$$

$$\dot{\varphi}_{cl} = \sqrt{V\varepsilon} = H M_{pl}\sqrt{\varepsilon}$$

$$\Phi_i = \frac{H\varphi}{H M_{pl}\sqrt{\varepsilon}} = \frac{\varphi}{M_{pl}\sqrt{\varepsilon}}$$ (classical random variable)

$$\underbrace{\langle \Phi_i \Phi_i \rangle}_{\text{width of distr.}} = \frac{1}{M_{pl}^2\,\varepsilon}\langle \varphi\varphi\rangle =$$

$$= \frac{H^2}{\varepsilon M_{pl}^2} \frac{1}{k^3} \delta(\vec{k} - \vec{k}')$$

$$\frac{H^2}{\varepsilon M_{pl}^2} \simeq 10^{-10} \quad \text{to match observations}$$

$\varepsilon \sim 10^{-2}$ $\Longrightarrow$ $\frac{H}{M_{pl}} \sim 10^{-6}$

$H \ll M_{pl}$

in our approx. (exact de Sitter) we got exact $\frac{1}{k^3}$, if slow-roll corrections are included, one gets

$$P(k) = \frac{1}{k^{3+1-n_s}}$$

for our simple model $n_s - 1 = 2\eta - 6\varepsilon$

Experimentally $n_s = 0.965 \pm 0.004$

consistent with $\varepsilon, \eta \sim 10^{-2}$ ✓